

Small x QCD and Multigluon States: a Color Toy Model

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Abstract

We introduce and study a toy model with a finite number of degrees of freedom whose Hamiltonian presents the same color structure of the BKP system appearing in the studies of QCD in the Regge limit. We address within this toy model the question of the importance of finite N_c corrections with respect to the planar limit case.

1 Introduction

The large N_c expansion [1] is a widely popular framework of approximations which has been successfully applied to gauge theories and has given at leading order some analytical results otherwise impossible to obtain. Within the Regge limit of QCD scattering amplitudes, L.N. Lipatov found [2] that systems of reggeized gluons evolving in rapidity in the leading logarithmic approximation (LLA) were showing the emergence of an integrable structure in the planar limit. Similar features were found later in other kinematical regimes for other QCD observables. Moreover the $N = 4$ SYM theory has been investigated at different order in perturbation theory and is now believed to be integrable at all orders.

But if one considers some QCD observables at the physical point $N_c = 3$ the situation is much more complicated and even the order of the corrections with respect to the planar limit are not really known. This is the situation, for example, for the spectrum of the BKP kernel [3, 4] at one loop, which describes the high energy behavior in the Regge limit of a system of reggeized gluons.

It is the purpose of this talk to discuss a toy model [5] which has a color structure similar to the BKP system but a different “configuration” dynamics with a finite number of d.o.f., constrained only by the fact that the two Hamiltonians must have the same leading eigenvalues in the large N_c limit for both one and two cylinder topologies. The main motivation to study this model is to understand in a simpler case how much the large N_c approximation fails to reproduce the dynamics at finite N_c . In order to understand this we shall study the spectrum of such model as a function of N_c .

2 Small x QCD: the LLA BKP kernel

Let us start by giving a brief overview of the LLA kernels encoding the evolution in rapidity of systems of interacting reggeized gluons, which provide a convenient perturbative description of some relevant QCD degrees of freedom in the Regge limit (small x). Their dynamics determine the high energy behavior of the cross sections, typically associated to the so called BFKL (perturbative) pomeron [6, 7]. In the simplest form, the BFKL pomeron turns out to be a composite

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state of two interacting reggeized gluons “living” in the transverse configuration plane in the colorless configuration. Its kernel or Hamiltonian is infrared finite and in LLA is constructed summing the perturbative contributions of different Feynman diagrams: in particular the virtual ones (reggeized one loop gluon trajectories) ω and the real ones (associated to an effective real gluon emission vertex) V . One writes formally $H = \omega_1 + \omega_2 + \vec{T}_1 \vec{T}_2 V_{12}$ where \vec{T}_i are the generators of the color group in adjoint representation. In the colorless case one has $\vec{T}_1 \vec{T}_2 = -N_c$ and finally one obtains:

$$H_{12} = \ln |p_1|^2 + \ln |p_2|^2 + \frac{1}{p_1 p_2^*} \ln |\rho_{12}|^2 p_1 p_2^* + \frac{1}{p_1^* p_2} \ln |\rho_{12}|^2 p_1^* p_2 - 4\Psi(1), \quad (1)$$

where $\Psi(x) = d \ln \Gamma(x)/dx$, a factor $\bar{\alpha}_s = \alpha_s N_c / \pi$ has been omitted and the gluon holomorphic momenta and coordinates have been introduced. One has the freedom, because of gauge invariance to choose a description within the Möbius space [8–10]. Then the BFKL hamiltonian has the property of the holomorphic separability ($H_{12} = h_{12} + \bar{h}_{12}$) and is invariant under the Möbius group $SL(2, C)$ transformations, whose generators for the holomorphic sector in the Möbius space for the principal series of unitary representations are given by $M_r^3 = \rho_r \partial_r$, $M_r^+ = \partial_r$, $M_r^- = -\rho_r^2 \partial_r$. The associated Casimir operator for two gluons is $M^2 = |\vec{M}|^2 = -\rho_{12}^2 \partial_1 \partial_2$ where $\vec{M} = \sum_{r=1}^2 \vec{M}_r$ and $\vec{M}_r \equiv (M_r^+, M_r^-, M_r^3)$. Note that, after defining formally $J(J-1) = M^2$, one may write $h_{12} = \psi(J) + \psi(1-J) - 2\psi(1)$.

The eigenstates and eigenvalues of the full hamiltonian in eq. (1), $H_{12} E_{h, \bar{h}} = 2\chi_h E_{h, \bar{h}}$ are labelled by the conformal weights $h = \frac{1+n}{2} + i\nu$, $\bar{h} = \frac{1-n}{2} + i\nu$. The leading eigenvalue, at the point $n = \nu = 0$, has a value $\chi_{max} = 4 \ln 2 \approx 2.77259$, responsible for the rise of the total cross section as $s^{\bar{\alpha}_s \chi_{max}}$, which corresponds to a strong violation of unitarity.

Let us now consider the evolution in rapidity of composite states of more than 2 reggeized gluons [3, 4]. The BKP Hamiltonian in LLA, acting on a colorless state, can be written in terms of the BFKL pomeron Hamiltonian and has the form (see [2])

$$H_n = -\frac{1}{N_c} \sum_{1 \leq k < l \leq n} \vec{T}_k \vec{T}_l H_{kl}. \quad (2)$$

This Hamiltonian is conformal invariant but can be solved only for 3 reggeized gluons, since the color structure factorizes, leaving an integrable dynamics [2]. Different families of odderon solutions were found [11, 12]. The family of solutions given in [12] are the leading ones corresponding to intercept up to 1 and have a non null coupling to photon-meson impact factors [13].

The case of more than three reggeized gluon is in general not solvable but in the large N_c limit, taking the one cylinder topology (1CT), one obtains the integrable Hamiltonian

$$H_n^\infty = \frac{1}{2} [H_{12} + H_{23} + \dots + H_{n1}] = h_n + \bar{h}_n, \quad (3)$$

i.e. there exists a set of other $n-1$ operators q_r , which commute with it and are in involution. This integrable model is a non compact generalization of the Heisenberg XXX spin chains [2, 14] and has been intensively studied with different techniques in the last decade [15–20].

Here we remind the value of the highest eigenvalue of a system of 4 reggeized gluons: $H_4^\infty \psi_4 = 2E_4^{1CT} \psi_4$. The maximum value found, for zero conformal spin, is

$$E_4^{1CT} = 0.67416. \quad (4)$$

How to go beyond the large N_c approximation is not an easy question to answer. One may be tempted to apply variational or perturbative techniques to the spectral problem, which nevertheless appears to be quite involved.

3 Color structure for the 4 gluon case

Let us analyze the color structure of the BKP kernel H_4 for four gluons, given in eq. (2). It is acting on 4-gluon states, which may be represented as functions of the transverse plane coordinates and of the gluon colors $v(\{\rho_i\})^{a_1 a_2 a_3 a_4}$. Since the four gluons are in a total color singlet the color vector $v^{a_1 a_2 a_3 a_4}$ can be described in terms of the color state of a two gluon subchannel. On such a subspace, introducing the projectors $P[R_i]_{a_1 a_2}^{a'_1 a'_2}$ onto irreducible representations of $SU(N_c)$, one has $1 = P_1 + P_{8A} + P_{8S} + P_{10+\bar{1}0} + P_{27} + P_0 = \sum_i P[R_i]$, where $Tr P[R_i] = d_i$ is the dimension of the corresponding representation and we consider a unique subspace for the 10 and $\bar{1}0$ representations. This is convenient for our purposes and we shall therefore work with 6 different projectors to span the color space of two gluons.

On considering gluons (1, 2) to be the reference channel we introduce as the base for the color vector space the set $\{P[R_i]_{a_1 a_2}^{a_3 a_4}\}$ of projectors and write

$$v^{a_1 a_2 a_3 a_4} = \sum_i v^i (P[R_i]_{a_1 a_2}^{a_3 a_4}) \quad \text{or} \quad v = \sum_i v^i P_{12}[R_i]. \quad (5)$$

Having chosen a color basis, the next step is to write the BKP kernel with respect to it. Since $\sum_i \vec{T}_i v = 0$ one may finally obtain:

$$H_4 = -\frac{1}{N_c} \left[\vec{T}_1 \vec{T}_2 (H_{12} + H_{34}) + \vec{T}_1 \vec{T}_3 (H_{13} + H_{24}) + \vec{T}_1 \vec{T}_4 (H_{14} + H_{23}) \right]. \quad (6)$$

Let us now write explicitly the action of the color operators $\vec{T}_i \vec{T}_j = \sum_a T_i^a T_j^a$ which are associated to the interaction between the gluons labelled i and j . We start from the simple “diagonal channel” for which we have relation $\vec{T}_i \vec{T}_j = -\sum_k a_k P_{ij}[R_k]$ with coefficients $a_k = (N_c, \frac{N_c}{2}, \frac{N_c}{2}, 0, -1, 1)$. Consequently we can write in the (1, 2) reference base

$$(\vec{T}_1 \vec{T}_2 v)^j = -a_j v^j = -(A v)^j, \quad (7)$$

where $A = \text{diag}(a_k)$. The action on v of the $\vec{T}_1 \vec{T}_3$ and $\vec{T}_1 \vec{T}_4$ operators is less trivial and is constructed in terms of the $6j$ symbols of the adjoint representation of $SU(N_c)$ group:

$$(\vec{T}_1 \vec{T}_3 v)^j = -\sum_i \left(\sum_k C_k^j a_k C_i^k \right) v^i = -(C A C v)^j \quad (8)$$

and

$$(\vec{T}_1 \vec{T}_4 v)^j = -\sum_i \left(\sum_k s_j C_k^j a_k C_i^k s_i \right) v^i = -(S C A C S v)^j. \quad (9)$$

The matrix C is the symmetric crossing matrix build on the $6j$ symbols and $S = \text{diag}(s_j)$ is constructed on the parities $s_j = \pm 1$ of the different representations R_j .

We can therefore write the general BKP kernel for a four gluon state, given in eq. (6), as

$$H_4 = \frac{1}{N_c} [A(H_{12} + H_{34}) + CAC(H_{13} + H_{24}) + SCACS(H_{14} + H_{23})] \quad (10)$$

One can check that if we make trivial the transverse space dynamics, replacing the H_{ij} operators by a unit operators, the general BKP kernel in eq. (2) becomes $H_n = \frac{n}{2}\hat{1}$ and indeed one can verify that $A + CAC + SCACS = N_c\hat{1}$.

Let us make few considerations on the large N_c limit approximation. As we have already discussed, in the Regge limit one faces the factorization of an amplitude in impact factors and a Green's function which exponentiates the kernel. The topologies resulting from the large N_c limit depend on the impact factor structure. In particular one expects the realization of two cases: the one and two cylinder topologies. The former corresponds to the case, well studied, of the integrable kernel, Heisenberg XXX spin chain-like. It is encoded in the relation: $\vec{T}_i \vec{T}_j \rightarrow -\frac{N_c}{2} \delta_{i+1,j}$ which leads to $H_4 = \frac{1}{2}(H_{12} + H_{23} + H_{34} + H_{41})$. It is characterized by eigenvalues corresponding to an intercept less than a pomeron. The latter case instead is expected to have a leading intercept, corresponding to an energy dependence given by two pomeron exchange. Consequently one expects at finite N_c a contribution with an energy dependence even stronger. In the two cylinder topology the color structure is associated to two singlets ($\delta_{a_1 a_2} \delta_{a_3 a_4}$, together with the other two possible permutations). Such a structure is indeed present in the analysis, within the framework of extended generalized LLA, of unitarity corrections to the BFKL pomeron exchange [21] and diffractive dissociation in DIS [22], where the perturbative triple pomeron vertex (see also [23, 24]) was discovered and shown to couple exactly to the four gluon BKP kernel.

It is therefore of great importance to understand how much the picture derived in the planar $N_c = \infty$ case is far from the real situation with $N_c = 3$. One clearly expects for example that the first corrections to the eigenvalues of the BKP kernel are proportional to $1/N_c^2$, but what is unknown is the multiplicative coefficient as well as the higher order terms.

4 A BKP toy model

In this section we shall consider a toy model [5], different from the BKP system, but sharing several features with it and analysis within it if the large N_c approximation might be more or less satisfactory.

Besides the color space, a state of n reggeized gluons undergoing the BKP evolution belongs to the configuration space R^{2n} , associated to the position or momenta in the transverse plane of the n gluons. The operators H_{kl} act (see eq. (10)) on such a state and, on the Möbius space, can be written in terms of the Casimir of the Möbius group, i.e. in terms of the scalar product of the generators of the non compact spin group $SL(2, C)$: $H_{kl} = H_{kl}(\vec{M}_k \cdot \vec{M}_l)$.

We are therefore led to consider a class of toy models where the BKP configuration space R^{2n} is substituted by the space V_s^n where V_s is the finite space spanned by spin states belonging to the irreducible representation of $SU(2)$ with spin s . In particular we shall consider quantum

systems with an Hamiltonian:

$$\mathcal{H}_n = -\frac{1}{N_c} \sum_{1 \leq k < l \leq n} \vec{T}_k \vec{T}_l f(\vec{S}_k \vec{S}_l), \quad (11)$$

where \vec{S}_i are $SU(2)$ generators associated to the particle i in any chosen representation and f is a generic function. A particular toy model is therefore specified by giving the spin s of each particle (“gluons”) and the function f . Our BKP toy model is built choosing the spin $s = 1$ case in a global singlet state v ($\sum_i \vec{S}_i v = 0$) and the family of functions f

$$f_\alpha(x) = 2\text{Re} \left[\psi \left(\frac{1}{2} + \sqrt{-\alpha(4+2x)} \right) \right] - 2\psi(1). \quad (12)$$

This form is suggested by the conformal spin $n = 0$ BFKL Hamiltonian with the substitution $\frac{1}{4} + L_{ij}^2 \rightarrow -\alpha S_{ij}^2$ which assures to have the same leading eigenvalue for any α , since both expressions have the value zero as upper bound. The parameter α will be chosen in order to constrain the full 4-particle Hamiltonian (11) to have the same leading eigenvalue as the QCD BKP system in the large N_c limit an one cylinder topology (at zero conformal spin), given in eq. (4). This “BKP toy model” will be used to investigate finite N_c effects.

Since we have chosen to work with states singlet under $SU(2)_{spin\ conf}$ also for the spin part we employ the 2 particle subchannel decomposition in irreducible representations, in a way similarly adopted for the color part. After that one is left with the problem of diagonalizing an Hamiltonian which is a matrix 18×18 . Therefore we proceed by introducing for 2 particle spin 1 states the resolution of unity $1 = Q_1 + Q_3 + Q_5 = \sum_i Q[R_i]$ which let us write $f(\vec{S}_i \vec{S}_j) = \sum_k f(-b_k) Q_{ij}[R_k]$ with $b_k = (2, 1, -1)$. using a power series representation ($Q_{ij}[R_k]$ are projectors). Introducing the crossing matrices D and the parity matrix S' we obtain the relations $(f(\vec{S}_1 \vec{S}_2) v)^j = (B v)^j$, $(f(\vec{T}_1 \vec{T}_3) v)^j = (D B D v)^j$ and $(f(\vec{T}_1 \vec{T}_4) v)^j = (S' D B D S' v)^j$. It is then straightforward to derive a matrix form for the Hamiltonian of this toy model

$$\mathcal{H}_{4a} = \frac{2}{N_c} (A \otimes B + C A C \otimes D B D + S C A C S \otimes S' D B D S') \quad (13)$$

which depends on N_c and on the parameter α through the function f_α given in eq. (12).

In the large N_c limit one faces for the Hamiltonian two possible cases (see [5] for more details): the one cylinder topology (1CT) which corresponds to the simpler Hamiltonian $\mathcal{H}_{4a}^{1CT} = B + S' D B D S'$ and the two cylinder topology (2CT) corresponding to the even simpler Hamiltonian $\mathcal{H}_{4a}^{2CT} = 2B$. Let us remark that while in the case of $N_c > 3$ we consider a basis for the vector states made of $P[R_i]Q[R_j]$ with 18 elements since in the color sector there is also the P_0 projector, the case $N_c = 3$ is characterized by a basis of 15 elements.

The last step to obtain the BKP toy model is to fix the parameter α by requiring \mathcal{H}_{4a}^{1CT} to have the value of eq. (4) so we obtain $\alpha = 2.80665$. We are therefore left with an Hamiltonian which is just a function of the number of colors N_c .

Let us now consider its spectrum for the cases $N_c = 3$ and $N_c = \infty$. Here we report just the leading eigenvalues of \mathcal{H}_{4a} with their multiplicities: $(7.042, 2 \times 5.519, 2 \times 1.123, \dots)$. Changing N_c from 3 to ∞ we observe that the first three move to the 2CT leading eigenvalue 5.545

while the next two move to the 1CT leading eigenvalue 0.674. With very good approximation one finds that the N_c dependence of the leading eigenvalue E_0 is given by

$$E_0(N_c) = E_0(\infty) \left(1 + \frac{2.465}{N_c^2} \right). \quad (14)$$

One can see that for this toy model the large N_c approximation corresponds to an error of about 27%, an error which is not negligible because the coefficient of the leading correction to the asymptotic value, proportional to $1/N_c^2$, is a large number. The color- “spin” configuration mixing which is encoded in the eigenvectors has been also studied.

5 Conclusions

We have introduced a family of dynamical models describing interacting particles with color and spin degrees of freedom in order to see how much the large N_c approximation is significant when one is trying to extract the spectrum of these quantum systems. In particular we have investigated a toy model, constructed to mimic some features of the 4 gluon BKP system. We have determined the N_c dependence of the spectrum and discussed the $N_c = \infty$ limit finding for the leading eigenvalue corrections of about 30% at $N_c = 3$.

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